

Completeness of the set of scattering amplitudes ^{*†‡}

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Abstract

Let $f \in L^2(S^2)$ be an arbitrary fixed function on the unit sphere S^2 , with a sufficiently small norm, and $D \subset \mathbb{R}^3$ be an arbitrary fixed bounded domain. Let $k > 0$ and $\alpha \in S^2$ be fixed.

It is proved that there exists a potential $q \in L^2(D)$ such that the corresponding scattering amplitude $A(\alpha') = A_q(\alpha') = A_q(\alpha', \alpha, k)$ approximates $f(\alpha')$ with arbitrary high accuracy: $\|f(\alpha') - A_q(\alpha')\|_{L^2(S^2)} \leq \varepsilon$, where $\varepsilon > 0$ is an arbitrarily small fixed number. The results can be used for constructing nanotechnologically "smart materials".

1 Introduction

Let $D \subset \mathbb{R}^3$ be a bounded domain, $k = \text{const}$, $\alpha \in S^2$, S^2 is the unit sphere. Consider the scattering problem:

$$[\nabla^2 + k^2 - q(x)]u = 0 \quad \text{in} \quad \mathbb{R}^3, \quad (1)$$

$$u = u_0 + A_q(\alpha', \alpha, k) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad u_0 = e^{ik\alpha \cdot x}, \quad r = |x| \rightarrow \infty, \quad \alpha' = \frac{x}{r}, \quad (2)$$

The coefficient A_q is called the scattering amplitude, and $q \in L^2(D)$ is a potential. The solution to (1)-(2) is called the scattering solution. It solves the equation

$$u = u_0 - Tu, \quad Tu := \int_D g(x, y) q(y) u(y) dy, \quad (3)$$

$$g = g(x, y, k) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad u = u(y) = u(y, \alpha, k).$$

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The solution u to (3) is unique in $L^2(D)$ for any complex-valued q for which $\|T\| < 1$, i.e., for any sufficiently small q .

We are interested in the following problem, which differs from the standard inverse scattering problem with fixed-energy data, studied in [R].

Question (Problem) P. *Given an arbitrary fixed $f(\alpha') \in L^2(S^2)$, can one find a $q \in L^2(D)$, such that*

$$\|A_q(\alpha') - f(\alpha')\|_{L^2(S^2)} \leq \varepsilon, \quad A_q(\alpha') = A_q(\alpha', \alpha, k), \quad (4)$$

where $\varepsilon > 0$ is an arbitrary small number, $\alpha \in S^2$ and $k > 0$ are fixed?

The answer to this question was not known. The scattering problem in (1)–(2) has been studied much (see, e.g., [C], [P], [R]). The inverse scattering problem with fixed-energy data (ISP) was solved in [R1], [R2], [R3], and [R, Chapter 5]. This problem consists of finding $q \in L^2(D)$ from the corresponding scattering amplitude $A_q(\alpha', \alpha, k)$ given for all $\alpha', \alpha \in S^2$ at a fixed $k > 0$. It was proved in [R1] that this problem has a unique solution. In [R3] reconstruction algorithms were proposed for finding q from exact and from noisy data, and stability estimates were established for the solution.

The problem P we have posed in this paper is different from the ISP. The data $A_q(\alpha')$ do not determine $q \in L^2(D)$ uniquely, in general. The potential q for which (4) holds is not unique even if $\varepsilon = 0$ and $f(\alpha') = A_q(\alpha', \alpha)$ for some $\alpha \in S^2$ and $q \in L^2(D)$. We want to know if there is a $q \in L^2(D)$, $q = q_\varepsilon(x)$, such that (4) holds with an arbitrarily small fixed $\varepsilon > 0$.

We prove that the answer is yes, provided that f is sufficiently small. The "smallness" condition will be specified in our proof. The question itself is motivated by the problem P_1 studied in [R4]:

Problem P_1 . *Can one distribute small acoustically soft particles in a bounded domain so that the resulting domain would have a desired radiation pattern, i.e., a desired scattering amplitude?*

The problem P, which is studied here, is of independent interest.

The following two lemmas allow us to give a positive answer to Question P under the "smallness" assumption.

Lemma 1. *Let $f \in L^2(S^2)$ be arbitrary and $k > 0$ be fixed. Then*

$$\inf_{h \in L^2(D)} \|f(\alpha') + \frac{1}{4\pi} \int_D e^{-ik\alpha' \cdot x} h(x) dx\|_{L^2(S^2)} = 0. \quad (5)$$

Lemma 2. *Let $h \in L^2(D)$ be arbitrary, with a sufficiently small norm. Then*

$$\inf_{q \in L^2(D)} \|h - qu\|_{L^2(D)} = 0, \quad (6)$$

where $u = u(x, \alpha, k)$ is the scattering solution corresponding to q . Under the above "smallness" assumption, there exists a potential q , such that $qu = h$.

If Lemmas 1 and 2 are proved, then the positive answer to Question P (under the "smallness" assumption) follows from the well-known formula for the scattering amplitude:

$$A_q(\alpha') = -\frac{1}{4\pi} \int_D e^{-ik\alpha' \cdot x} q(x) u(x, \alpha, k) dx, \quad (7)$$

in which $k > 0$ and $\alpha \in S^2$ are fixed. The answer is given in Theorem 1.

Theorem 1. *Let $\varepsilon > 0$, $k > 0$, $\alpha \in S^2$ and $f \in L^2(S^2)$ be arbitrary, fixed, with a sufficiently small norm. Then there is a $q \in L^2(D)$ such that (4) holds.*

There are many potentials q for which (4) holds.

In Section 2 proofs are given.

In Section 3 a method is given for finding $q \in L^2(D)$ such that (4) holds.

2 Proofs

Proof of Lemma 1. If (5) is not true, then there is an $f \in L^2(S^2)$ such that

$$0 = \int_{S^2} d\beta f(\beta) \int_D e^{-ik\beta \cdot x} h(x) dx = \int_D dx h(x) \int_{S^2} f(\beta) e^{-ik\beta \cdot x} d\beta \quad \forall h \in L^2(D). \quad (8)$$

Since $h(x)$ is arbitrary, relation (8) implies

$$\int_{S^2} f(\beta) e^{-ik\beta \cdot x} d\beta = 0 \quad \forall x \in D. \quad (9)$$

The left-hand side of (9) is the Fourier transform of a compactly supported distribution $\frac{\delta(\lambda-k)}{k^2} f(\beta)$, where $\delta(\lambda-k)$ is the delta-function. Since the Fourier transform is injective, it follows that $f(\beta) = 0$. This proves Lemma 1.

An alternative proof can be given. It is known that

$$e^{-ik\beta \cdot x} = \sum_{\ell=0, -\ell \leq m \leq \ell}^{\infty} 4\pi i^\ell j_\ell(kr) \overline{Y_{\ell,m}(-x^0)} Y_{\ell,m}(\beta), \quad r := |x|, \quad x^0 := \frac{x}{r}, \quad (10)$$

where $Y_{\ell,m}$ are orthonormal in $L^2(S^2)$ spherical harmonics, $Y_{\ell,m}(-x^0) = (-1)^\ell Y_{\ell,m}(x^0)$, $j_\ell(r) := (\frac{\pi}{2r})^{1/2} J_{\ell+\frac{1}{2}}(r)$, and $J_\ell(r)$ is the Bessel function. Let

$$f_{\ell,m} := (f, Y_{\ell,m})_{L^2(S^2)}. \quad (11)$$

From (9) and (10) it follows that

$$f_{\ell,m} j_\ell(kr) = 0, \quad \forall x \in D, -\ell \leq m \leq \ell. \quad (12)$$

If $k > 0$ is fixed, one can always find $r = |x|$, $x \in D$, such that $j_\ell(kr) \neq 0$.

Thus, (12) implies $f_{\ell,m} = 0 \quad \forall \ell, -\ell \leq m \leq \ell$. Lemma 1 is proved. The "smallness" assumption is not needed in this proof. \square

Proof of Lemma 2. In this proof we use the "smallness" assumption. If the norm of f is sufficiently small, then the norm of h is small so that condition (23) (see below) is satisfied. If this condition is satisfied, then formula (24) (see below) yields the desired potential q , and $h = qu$, where u is the scattering solution, corresponding to q . Therefore, the infimum in (6) is attained. Lemma 2 is proved. \square

Let us give another argument, which shows the role of the "smallness" assumption from a different point of view. Note, that if $\|q\| \rightarrow 0$, then the set of the functions qu is a linear set. In this case, if one assumes that (6) is not true, then one can claim that there is an $h \in L^2(D)$, $h \neq 0$, such that

$$\int_D dx h(x) q(x) u(x) = 0 \quad \forall q \in L^2(D), \quad \|q\| \ll 1. \quad (13)$$

Choose

$$q = c \bar{h} e^{-ik\alpha \cdot x}, \quad c = \text{const} > 0. \quad (14)$$

Let c be so small that $\|T\| = O(c) < 1$, where $T : L^2(D) \rightarrow L^2(D)$ is defined in (3). Then equation (3) is uniquely solvable and

$$u = u_0 + O(c) \quad \text{as } c \rightarrow 0. \quad (15)$$

From (13) and (15) one gets

$$c \int_D |h|^2 dx + O(c^2) = 0 \quad \forall c \in (0, c_0). \quad (16)$$

If $c \rightarrow 0$, then (16) implies $\int_D |h(x)|^2 dx = 0$. Therefore $h = 0$.

3 A method for finding q for which (4) holds

Lemmas 1 and 2 show a method for finding a $q \in L^2(D)$ such that (4) holds. Given $f(\alpha') \in L^2(S^2)$, let us find $h(x)$ such that

$$\|f(\alpha') + \frac{1}{4\pi} \int_D e^{-ik\alpha' \cdot x} h(x) dx\|_{L^2(S^2)}^2 < \varepsilon^2. \quad (17)$$

This is possible by Lemma 1. It can be done numerically by taking $h = h_n = \sum_{j=1}^n c_j \varphi_j(x)$, where $\{\varphi_j\}_{1 \leq j < \infty}$ is a basis of $L^2(D)$, and minimizing the quadratic form on the left-hand side of (17) with respect to c_j , $1 \leq j < n$. For sufficiently large n the minimum will be less than ε^2 by Lemma 1.

The minimization with the accuracy ε^2 can be done analytically: let B_b be a ball of radius b centered at a point $0 \in D$, $B_b \subset D$, $h = 0$ in $D \setminus B_b$,

$$h(x) = \sum_{\ell=0, -\ell \leq m \leq \ell}^{\infty} h_{\ell, m}(r) Y_{\ell, m}(x^0) \quad \text{in } B_b,$$

$$f(\alpha') = \sum_{\ell=0, -\ell \leq m \leq \ell}^{\infty} f_{\ell,m} Y_{\ell,m}(\alpha'),$$

and

$$\sum_{l>L, -\ell \leq m \leq \ell} |f_{\ell,m}|^2 < \varepsilon^2.$$

For $0 \leq \ell \leq L$ let us equate the Fourier coefficients of $f(\alpha')$ and of the integral from (17). We use the known formula:

$$e^{-ik\beta \cdot x} = \sum_{\ell=0, -\ell \leq m \leq \ell} 4\pi i^\ell j_\ell(kr) \overline{Y_{\ell,m}(-x^0)} Y_{\ell,m}(\beta).$$

This leads to the following relations for finding $h_{\ell,m}(r)$:

$$f_{\ell,m} = -(-i)^\ell \sqrt{\frac{\pi}{2k}} \int_0^b r^{3/2} J_{\ell+\frac{1}{2}}(kr) h_{\ell,m}(r) dr, \quad 0 \leq \ell \leq L. \quad (18)$$

There are many $h_{\ell,m}(r)$ satisfying equation (18). Let us take $b = 1$ and use [B, Formula 8.5.5],

$$\begin{aligned} & \int_0^1 x^{\mu+\frac{1}{2}} J_\nu(kx) dx \\ &= k^{-\mu-\frac{3}{2}} \left[\left(\gamma + \mu - \frac{1}{2} \right) k J_\nu(r) S_{\mu-\frac{1}{2}, \nu-1}(k) - k J_{\nu-1}(k) S_{\mu+\frac{1}{2}, \nu}(k) + 2^{\mu+\frac{1}{2}} \frac{\Gamma\left(\frac{\mu+\nu}{2} + \frac{3}{4}\right)}{\Gamma\left(\frac{\nu-\mu}{2} + \frac{1}{4}\right)} \right] \\ &:= g_{\mu,\nu}(k), \end{aligned}$$

where $S_{\mu,\nu}(k)$ are Lommel's functions. Thus, one may take

$$h_{\ell,m}(r) = \frac{f_{\ell,m}}{-(-i)^\ell \sqrt{\frac{\pi}{2k}} g_{1,\ell+\frac{1}{2}}(k)}, \quad 0 \leq \ell \leq L; \quad h_{\ell,m}(r) = 0 \quad \ell > L.$$

Thus, the coefficients $h_{\ell,m}(r)$ do not depend on $r \in (0, 1)$ in this choice of h : inside the ball B_b the function $h(x)=h(x^0)$ depends only on the angular variables, and outside this ball $h = 0$.

With this choice of $h_{\ell,m}(r)$ the left-hand side of (17) equals to

$$\sum_{\ell=L+1, -\ell \leq m \leq \ell}^{\infty} |f_\ell|^2 < \varepsilon^2.$$

The above choice of $h_{\ell,m}(r)$, which yields an analytical choice of $h(x)$, is one of the choices for which (17) holds. The function $g_{1,\ell+\frac{1}{2}}(k)$ in the definition of $h_{\ell,m}(r)$ decays rapidly when ℓ grows. This makes it difficult numerically to calculate accurately $h_{\ell,m}(r)$ when ℓ is large. If $f = 1$ in a small solid angle and $f = 0$ outside of this angle, then one needs

large L to approximate f by formula (17) with small ε . Numerical difficulties arise in this case. This phenomenon is similar to the one known in optics and antenna synthesis as superresolution difficulties (see [R6], [R7], [R8]).

Let $h \in L^2(D)$ be a function for which (17) holds. Consider the equation

$$h = qu, \quad (19)$$

where $u = u(x; q)$ is the scattering solution, i.e., the solution to equation (3). Let

$$w := e^{-ik\alpha \cdot x} u, \quad G(x, y) := g(x, y) e^{-ik\alpha \cdot (x-y)}, \quad H := h e^{-ik\alpha \cdot x}, \quad \psi := qw. \quad (20)$$

Then equation (3) is equivalent to the equation

$$w = 1 - \int_D G(x, y) \psi(y) dy, \quad (21)$$

and $\psi(y) = H(y)$ by (19). Multiply (21) by q and get

$$\psi(x) = q(x) - q(x) \int_D G(x, y) \psi(y) dy.$$

Since $\psi(x) = H(x)$, we get

$$q(x) = H(x) \left[1 - \int_D G(x, y) H(y) dy \right]^{-1}. \quad (22)$$

If $H(x)$ is such that

$$\inf_{x \in D} \left| 1 - \int_D G(x, y) H(y) dy \right| > 0, \quad (23)$$

then $q(x)$, defined in (22), belongs to $L^2(D)$ and (4) holds. If H is sufficiently small then (23) holds. If D is small and H is fixed, then (23) holds because

$$\left| \int_D G(x, y) H dy \right| \leq \int_D \frac{|H| dy}{4\pi|x-y|} \leq \frac{1}{4\pi} \|H\|_{L^2(D)} \left(\int_D \frac{dy}{|x-y|^2} \right)^{1/2} = O(a^{1/2}) \|H\|,$$

where a is the radius of a smallest ball containing D .

Formula (22) can be written as

$$q(x) = h(x) \left[u_0 - \int_D g(x, y) h(y) dy \right]^{-1}, \quad (24)$$

where u_0 and g are defined in (3). Numerically equation (24) worked for $f(\beta)$ which were large in absolute value.

4 An idea of a method for making a material with the desired radiation pattern

Here we describe an idea of a method for calculation of a distribution of small particles, embedded in a medium, so that the resulting medium would have a desired radiation pattern for the plane wave scattering by this medium. This idea is described in more detail in [R4]. The results of this paper complement the results in [R4], but are completely independent of [R4], and are of independent interest.

Suppose that a bounded domain D is filled in by a homogeneous material. Assume that we embed many small particles into D . Smallness means that $k_0 a \ll 1$, where a is the characteristic dimension of a particle and k_0 is the wavenumber in the region D before the small particles were embedded in D . The question is:

Can the density of the distribution of these particles in D be chosen so that the resulting medium would have the desired radiation pattern for scattering of a plane wave by this medium?

For example, if the direction α of the incident plane wave is fixed, and the wave number k of the incident plane wave $e^{ik\alpha \cdot x}$ in the free space outside D is fixed, then can the particles be distributed in D with such a density that the scattering amplitude $A(\alpha', \alpha, k)$ of the resulting medium would approximate with the desired accuracy an a priori given arbitrary function $f(\alpha') \in L^2(S^2)$ in the sense (4)?

Assume that the particles are acoustically soft, that is, the Dirichlet boundary condition is assumed on their boundary, that $k_0 a \ll 1$ and $\frac{a}{d} \ll 1$, where $d > 0$ is the minimal distance between two distinct particles, and that the number N of the small particles tends to infinity in such a way that the limit $C(x) := \lim_{N \rightarrow \infty} \lim_{r \rightarrow 0} \frac{\sum_{D_j \subset B(x,r)} C_j}{|B(x,r)|}$ exists. Under these assumptions, it is proved in [R4] that the scattered field can be described by equation (1) with $q(x)$ that can be expressed analytically via $C(x)$. Here D_j is the region occupied by j -th acoustically soft particle, C_j is the electrical capacitance of the perfect conductor with the shape D_j , $B(x, r)$ is the ball of radius r centered at the point x , and $|B(x, r)| = \frac{4\pi r^3}{3}$ is the volume of this ball. If the small particles are identical, and \mathcal{C} is the electrical capacitance of one small particle, then $C(x) = \mathcal{N}(x)\mathcal{C}$, where $\mathcal{N}(x)$ is the number of small particles per unit volume around the point x . Formulas for the electrical capacitances for conductors of arbitrary shapes are derived in [R5].

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